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Linkages for x^{m}.

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By the following method of combining reciprocators (a Peaucellier cell with a modulus one) and bisectors (a pantograph with equal arms) a linkage for x^m may be obtained, m being any positive or negative integer or fraction.

Let the reciprocator be designated by R, the bisector by B, and the linkage for x^m by L^m. Then $L^{\frac{1}{m}}$ will designate the linkage for $x^{\frac{1}{m}}$, or the m^{th} root linkage, which may be obtained from that for x^m by transposing its arms.

If a and b be the distances of two points P_1 and P_2, measured in opposite directions, from a point O, or two points symmetrical with respect to O, and if the two poles of a B are made to coincide with P_1 and P_2, O' being the bisecting point, then it may be easily proved that $OO' = \frac{1}{2}(a - b)$.

Let $m = -n$, n being any positive integer or fraction. Then L^m may be obtained by combining L^n and one R, for $\frac{1}{x^n} = x^{-n} = x^m$.

Let $m = \frac{p}{q}$, p and q being positive integers. Then L^m may be obtained by combining L^p and $L^{\frac{1}{q}}$ for $(x^p)^{\frac{1}{q}} = x^{\frac{p}{q}} = x^m$.

Let $m = 2p - q$, p and q being integers and p positive. Then L^m may be obtained in two different ways by combining L^p, L^q, three R's and one B.

(a.) Add $-x^q$ and $+x^q$ to x^p, giving $x^p - x^q$ and $x^p + x^q$, and convert by two R's into $\frac{1}{x^p - x^q}$ and $\frac{1}{x^p + x^q}$. The B will then give

$$\frac{1}{2}\left[\frac{1}{x^p - x^q} - \frac{1}{x^p + x^q}\right] = \frac{x^q}{x^{2p} - x^{2q}}.$$

The remaining R gives $x^{2p-q} - x^q$, and adding x^q we have x^{2p-q} or x^m.

(b.) Subtract x^q from x^p, giving $x^p - x^q$. Convert by one R into $\frac{1}{x^p - x^q}$. By another R form $\frac{1}{x^p}$, and by the aid of the B subtract from $\frac{1}{x^p - x^q}$, giving $\frac{x^q}{2x^p(x^p - x^q)}$.

Fig. 1.

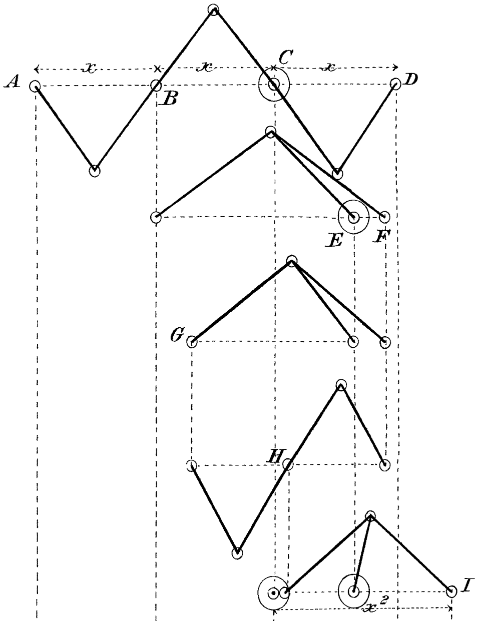


Fig. 2.

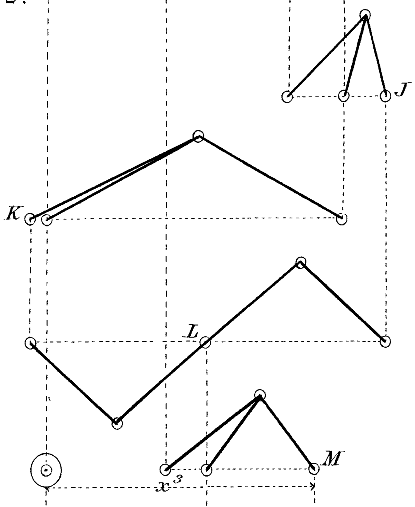


Fig. 3.

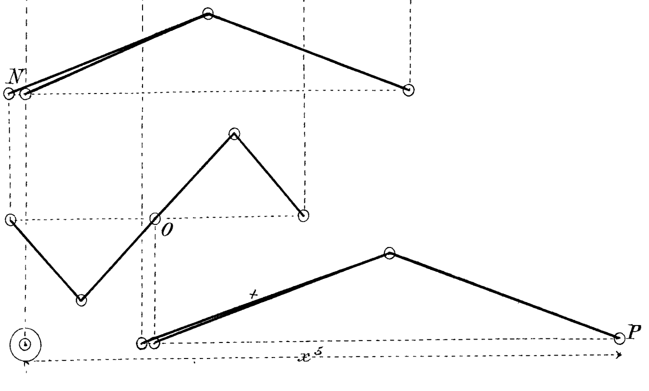


Fig. 4.

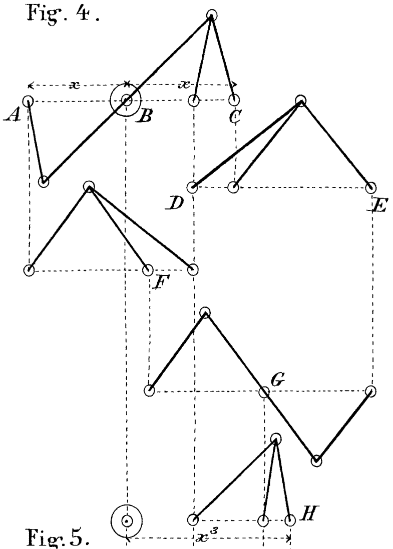


Fig. 5.

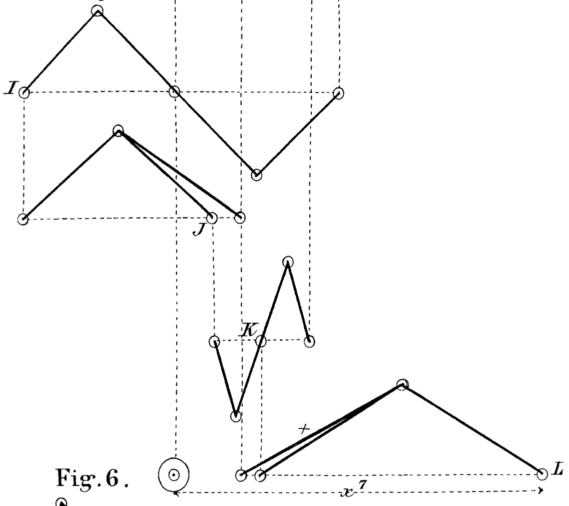
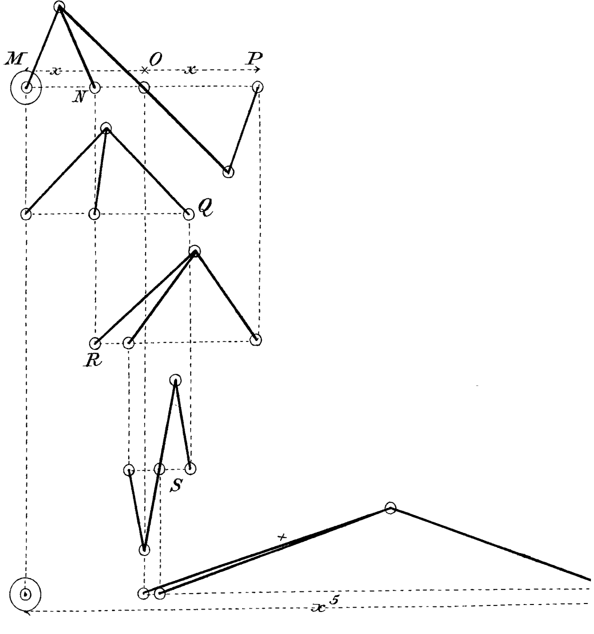


Fig. 6.



This may be converted into $x^{2p-q} - x^p$ by the last R, slightly modified, and adding x^p, we have x^{2p-q} or x^m.

The arrangement of the cells which renders it possible to add and subtract these various quantities will be best understood from the examples.

If $q = +1$, 0, or -1, $m = 2n - 1$, $2n$, or $2n + 1$; and as m must be of one of these forms we can always obtain L^m from L^n by either of the above methods. When m is odd, L^m may be made to depend upon the linkage for an odd or an even power. By making $n = 1$ we obtain $m = 1$, 2, or 3. Making $n = 2$ and 3 successively, we have $m = 3, 4, 5$; $5, 6, 7$, and so on, thus giving the linkages for every positive integral power.

In the figures the cells of the compound linkages have been separated for the sake of clearness, and only one half of each drawn. Points on the same vertical line should coincide. A small circle represents a joint, a large one a fixed point, and a heavy line a link. Each letter refers not only to the point in its immediate vicinity, but also to all points upon the same vertical line. A + or − sign indicates a positive or negative reciprocator.

Fig. 1 gives the linkage for x^2. Let $BC = CD = x$ and $CE = 1$. The portion AB and similar extensions are only added when x^5 and higher powers are to be derived by the first method. Then $BE = x + 1$ and $ED = x - 1$. By the two R's $GE = \frac{1}{x-1}$ and $EF = \frac{1}{x+1}$. By the B,

$$HE = \frac{1}{2}\left[\frac{1}{x-1} - \frac{1}{x+1}\right] = \frac{1}{x^2-1}.$$

The last R gives $EI = x^2 - 1$ and $CI = EI + 1 = x^2$. It is not necessary that E should be a fixed point, it is sufficient to have E and C connected by a link of a length unity.

Fig. 2 gives the derivation of the linkage for x^3 from that for x^2 by adding three R's and one B. $DI = x^2 - x$ and $IB = x^2 + x$. The two R's give $DJ = \frac{1}{x(x-1)}$ and $KB = \frac{1}{x(x+1)}$. The B gives $CL = \frac{1}{2}\left[\frac{1}{x(x-1)} - \frac{1}{x(x+1)}\right] = \frac{1}{x(x^2-1)}$. The last R gives $CM = x^3 - x$ and $BM = CM + x = x^3$. Let the two points C and E be connected by a link of a length unity, as before stated, and set free. Let B be made the fixed point, and we have a linkage in which the arm $BC = x$ and $BM = x^3$ measured in the same direction. It is necessary that B, C, and E should remain in the same straight line.

Fig. 3 gives the derivation of the linkage for x^5 from that for x^3 by adding two R's and one B. $AM = x^3 + x$, and the first R gives $AN = \frac{1}{x^3 + x}$,

$CL=\frac{1}{x^3-x}\cdot$ The B gives $BO=\frac{1}{2}\left[\frac{1}{x(x^2-1)}-\frac{1}{x(x^2+1)}\right]=\frac{1}{x(x^4-1)}\cdot$ The last R gives $BP=x^5-x$ and $AP=BP+x=x^5$. Setting free B and making A the fixed point, we have a linkage in which the arm $AB=x$ and $AP=x^5$ measured in the same direction.

Fig. 4 gives the derivation of the linkage for x^3 from those for x and $\frac{1}{x}$ by adding one B and three R's by the first method. $BC=x, BD=\frac{1}{x}\cdot$ $DC=\frac{x^2-1}{x}$ and the R gives $DE=\frac{x}{x^2-1}\cdot$ $DA=\frac{x^2+1}{x}$, and the R gives $DF=\frac{x}{x^2+1}\cdot$ The B gives $DG=\frac{1}{2}\left[\frac{x}{x^2-1}-\frac{x}{x^2+1}\right]=\frac{x}{x^4-1}\cdot$ The last R gives

$$DH=x^3-\frac{1}{x} \text{ and } HB=HD+\frac{1}{x}=x^3.$$

If the second method had been used, we should have arrived at the well-known form for the cube devised by Professor Sylvester, consisting of only three cells, and the reason is this, $x^p=x$, $x^q=x^{-1}$ $\therefore \frac{1}{x^p}=x^q$; and as x^q is already at command, having been formed by the first R, we do not need the second cell in the transformation; also, after this change we do not need the B when it is necessary to form $\frac{x}{x^2-1}-\frac{1}{x}$, thus giving three cells instead of five. See Fig. 6 and its explanation.

Fig. 5 gives the derivation of the linkage for x^7 from those for x^3 and $\frac{1}{x}$ by adding a B to transfer a distance, two R's and one B. The third R being unnecessary, as $DG=\frac{1}{x^3-\frac{1}{x}}$ had been formed in the linkage for x^3. $ID=x^3+\frac{1}{x}\cdot$ The R gives $JD=\frac{x}{x^4+1}\cdot$ The B gives $DK=\frac{1}{2}\left[\frac{x}{x^4-1}-\frac{x}{x^4+1}\right]=\frac{x}{x^8-1}\cdot$ The last R gives $DL=x^7-\frac{1}{x}$ and $BL=DL+\frac{1}{x}=x^7$.

Fig. 6 gives a form for x^5 consisting of five cells or thirty-two links. The first part of the transformation is conducted by the second method and the latter part by the first method. $MO=OP=x$. $MN=x-\frac{1}{x}\cdot$ The R gives $NQ=\frac{x}{x^2-1}\cdot$ $NO=\frac{1}{x}\cdot$ $OQ=\frac{x}{x^2-1}-\frac{1}{x}=\frac{1}{x(x^2-1)}\cdot$ (If this were inverted to a point Z to the right by a R, then ZM would equal x^3, thus giving the linkage for the cube devised by Professor Sylvester.) $NP=x+\frac{1}{x}\cdot$ The R gives $NR=\frac{x}{x^2+1}$ and $OR=\frac{1}{x}-\frac{x}{x^2+1}=\frac{1}{x(x^2+1)}\cdot$ The B then gives

$$OS = \frac{1}{2}\left[\frac{1}{x(x^2-1)} - \frac{1}{x(x^2+1)}\right] = \frac{1}{x(x^4-1)}.$$

The last R gives $OT = x^5 - x$ and $MT = OT + x = x^5$. Making M the fixed point, we have a linkage in which the arm $MO = x$ and $MT = x^5$ measured in the same direction.

It is unnecessary to give examples of the linkages for the higher powers, as their formation is a process as mechanical as writing out the successive terms of a series.

The above arrangements admit of a number of variations, some of which reduce the number of links.

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